STABILITY OF ROTATING FLOWS OF AN IDEAL FLUID

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We consider flows of an ideal incompressible fluid whose velocities in a cylindrical coordinate system (r, φ, z) have the form (0, V, W). We assume that V = V(r), W = W(r). We investigate the stability of the latter in a linear approximation. Such flows are idealizations of such natural phenomena as tornadoes and dust devils [1, 2]. Their investigation is important for an understanding of the role of instability in vortex breakdown [3]. Classical Couette flow between cylinders is also of this type.

At present there are few general theoretical results relating to the linear stability of rotating flows to nonaxisymmetric disturbances. This question was dealt with by Howard et al. [4], who obtained a representation of the normal oscillations in terms of a single equation and introduced an analog of the Richardson number. Leibovich [5] investigated the sufficient conditions for stability of such flows to axisymmetric disturbances. In [6, 7] stability to nonaxisymmetric disturbances was investigated numerically. A Poiseuille flow in a rotating tube was examined in [6], and a linear vortex in the wake of a wing in [7]. Several general results for flows with round streamlines (W = 0) were obtained in [8, 9], where the similarity of such flows to plane-parallel stratified flows was investigated. Questions of the stability of stratified flows are dealt with in [10, 11].

In this paper we obtain some general results relating to the stability of rotating flows to nonaxisymmetric disturbances. An estimate of the complex part of the disturbance spectrum is given (circle theorem). The problem with initial data is considered, and a rule for selection of the solution branch at the special point is formulated. The similarity and difference of rotating flows to flows with round streamlines and stratified flows are indicated.

1. Let (u, v, w) be the complex amplitudes of the velocity disturbance components corresponding to coordinates (r, φ, z). Let the disturbances have the form of normal waves; $u(r, \varphi, z, t) = \mathrm{Re}\{u(r)\exp[i(kz + m\varphi - \omega t)]\}$, etc. In [4] the linearized equations of motion and continuity were reduced to a single equation. For q = ur the latter can be written in the form

$$q'' + \left(\frac{1}{r} - \beta\right)q' - \frac{1}{r^2\rho}q + \left[\frac{A(r)}{\sigma} + \frac{G(r)}{\sigma^2}\right]q = 0, \tag{1.1}$$

where $\rho = 1/(m^2 + k^2r^2)$; $\beta = -\rho'/\rho$; $\sigma = V/r + kW/m - \omega/m$; $A(r) = (\beta\Omega + k\beta rW'/m - \Omega' - k(rW')'/m)/r$;

$$G(r) = \frac{2kV}{mr^2}\left(\frac{k}{m}\Omega r - W'\right);\quad \Omega = dV/dr + V/r.$$

(Below we consider flows between cylinders of radii R_1 and R_2.) Then to (1.1) we add the boundary conditions

$$q = 0 \text{ when } r = R_1,\ R_2. \tag{1.2}$$

For proof of the circle theorem we convert Eq. (1.1). We put q = σrf(r). After some algebra we obtain

$$(\rho r^3\sigma^2 f')' + 2\rho rLf - (m^2 + k^2r^2 - 1)\rho r\sigma^2 f - \rho' r^2(M^2 - (N - c)^2)f + 2\rho r^2 N'(N - c)f = 0, \tag{1.3}$$

where $c = \omega/m$; $M = V/r$; $N = kW/m$; $L = k^2rV\Omega/m^2$. We put $Q = \rho r^3|f'|^2 + \rho r(m^2 + k^2r^2 - 1)|f|^2$, $Q_1 = 2\rho r|f|^2$, $Q_2 = -\rho' r^2|f|^2$. We multiply (1.3) by f*, then integrate from R_1 to R_2, taking (1.2) into account. In the obtained equality we separate the imaginary and real parts:

$$2ic_i\left\{\int Q(M + N - c_r)\,dr + \int Q_2(N - c_r)\,dr - \int \frac{Q_1}{2} rN'\,dr\right\} = 0; \tag{1.4}$$

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$$\int Q\,[(M+N-c_r)^2-c_i^2]\,dr+\int Q_2\,[(N-c_r)^2-M^2-c_i^2]\,dr-\int Q_1 rN'(N-c_r)\,dr-\int Q_1 L\,dr=0. \quad (1.5)$$

We put $a=\min(M+N-rN'/2,\ M/2+N-rN'/4,\ M+N)$, $b=\max(M+N-rN'/2,\ M/2+N-rN'/4,\ M+N)$.

THEOREM. All the complex eigenvalues of Eq. (1.1) when $m^2>1$ satisfy the inequalities

$$a<c_r<b,$$

$$\left(c_r-\frac{a+b}{2}\right)^2+c_i^2<\left(\frac{b-a}{2}\right)^2+\max\left[V\left(\frac{k}{m}W'-\frac{k^2}{m^2}\Omega r\right),\,0\right].$$

Proof. Direct verification shows that for $m^2>1$ the inequality

$$Q\geqslant Q_1\geqslant Q_2\geqslant 0 \quad (1.6)$$

is fulfilled. We will prove the right-hand inequality in $a<c_r<b$. The left-hand one is proved in a similar way. We assume the opposite. Then, for (1.4), in view of (1.6), we have a chain of inequalities

$$\int Q\,(c_r-M-N)\,dr+\int Q_2\,(c_r-N)\,dr+\int\frac{Q_1}{2}rN'dr>$$
$$>\int(Q_1-Q_2)\left(c_r-M-N+\frac{rN'}{2}\right)dr+\int 2Q_2\left(c_r-\frac{M}{2}-N+\frac{rN'}{4}\right)dr>0.$$

Since $c_i\neq 0$ the last inequality contradicts (1.4). We make the the substitutions: $N_*=N-(a+b)/2$, $c_*=c-(a+b)/2$. Then the form of (1.4), (1.5) is not altered, except that N and c will be replaced by N_* and c_*. Using (1.4) we convert (1.5) to the form

$$\int Q\,[|c_*|^2-(M+N_*)^2]\,dr+\int Q_2\,(|c_*|^2+M^2-N_*^2)\,dr+\int Q_1\,(rN'N_*+L)\,dr=0. \quad (1.7)$$

We prove the last inequality of the theorem. We assume the opposite. Then $|c_*|^2>(M+N_*)^2$. In view of this and (1.6), for (1.7) we have the chain of inequalities

$$\int Q\,[|c_*|^2-(M+N_*)^2]\,dr+\int Q_2\,(|c_*|^2+M^2-N_*^2)\,dr+$$
$$+\int Q_1\,(rN'N_*+L)\,dr>\int(Q_1-Q_2)\,[|c_*|^2-(M+N_*)^2+rN'N_*+L]\,dr+$$
$$+\int Q_2\,[2|c_*|^2-(M+N_*)^2+M^2-N_*^2+rN'N_*+L]\,dr>$$
$$>\int(Q_1-Q_2)\left\{|c_*|^2-\left[\left(M+N_*-\frac{rN'}{2}\right)^2+V\left(\frac{k}{m}W'-\frac{k^2}{m^2}\Omega r\right)\right]\right\}+$$
$$+\int 2Q_2\left\{|c_*|^2-\left[\left(N_*+\frac{M}{2}-\frac{rN'}{4}\right)^2+\frac{V}{2}\left(\frac{k}{m}W'-\frac{k^2}{m^2}\Omega r\right)\right]\right\}>0,$$

which contradicts (1.7).

2. The proved proposition allows us to consider the problem with initial data. Subsequent results are obtained by methods similar to those in [5, 9-11]. Hence, the treatment will be brief with appropriate references.

We put the disturbance in the form $u(r,\varphi,z,t)=\mathrm{Re}\,\{u(r,t)\exp[i(kz+m\varphi)]\}$, $u(r,0)=u_0(r)$, etc. We put $\lambda=ru(p,r)$, where $u(p,r)=\int_0^\infty u(r,t)\exp(-pt)\,dt$ is the Laplace transform of the function u(r, t). From the linearized equations of motion and continuity we can obtain an equation for

$$\lambda''+\left(\frac{1}{r}-\beta\right)\lambda'-\frac{\lambda}{r^2\rho}+\left[\frac{A(r)}{\sigma}+\frac{G(r)}{\sigma^2}\right]\lambda=\frac{X_1}{\sigma}+\frac{X_2}{\sigma^2}. \quad (2.1)$$

Here the dash denotes differentiation with respect to r; $\sigma = V/r + kW/m - ip/m$; $X_1 = -i\cdot (ru_0)''/m - i(1/r - \beta)(ru_0)'/m$; $X_2 = 2kV(krv_0/m - w_0)/(mr)$.

We will assume that the function V/r + kW/m is monotonic. The results can be extended to the case of a nonmonotonic function as in [11]. Following [9-11], we find the asymptotic form of the function u(r, t) when $t \to \infty$. The analog of the Richardson number in this case is the number [4, 8]

$$J = \frac{2kV(k\Omega r/m - W')}{mr^2[(V/r)' + kW'/m]^2}.$$

We put $\nu(r) = \sqrt{1-4J}$ and consider disturbances belonging to the continuous spectrum. When $\nu > 0$, $k \neq 0$, $t \to \infty$ we have $u(r, t) \sim t^{\nu-1}\exp(-imUt)$, $U = V/r + kW/m$. We note that, as distinct from [9-11], for any r, where $V \neq 0$, $W' \neq 0$, there are k/m such that $J < 0$, $\nu > 1$. These values of k/m correspond to the most rapidly increasing solutions of the continuous spectrum. Thus, we can always find k/m for which disturbances of the radial velocity component of the rotating flow grow according to a power law, since there is always a point r where $V \neq 0$, $W' \neq 0$. The case $W' \equiv 0$, according to (1.1), (2.1) reduces simply to the case W = 0.

We again consider disturbances having the form of normal waves. Their behavior is described by Eq. (1.1). The latter has singularities at points $\sigma = 0$, i.e., at certain real values of ω. Solutions with $\sigma = 0$ are called singular neutral modes (SNM). We now follow [5, 9]. Let r_0 be such that $\sigma(r_0) = 0$. Then, in the vicinity of r_0 the solution of (1.1) has the form $X = AX_+ + BX_-$. Here $X = \sqrt{\rho r}\cdot q$; A and B are constants;

$$X_\pm = \xi^{1/2(1\pm\nu_0)}\Phi_\pm(r); \quad \nu_0 = \nu(r_0); \quad \xi = (r - r_0);$$

$\Phi_\pm$ are analytical functions of the form $\Phi_\pm = 1 + a_\pm\xi + \ldots$ in the vicinity of r_0. It follows from a consideration of the problem with initial data that SNM is the limit of solutions with Im $\omega > 0$ when Im $\omega \to 0$. This condition gives the rule for circumvention of the singularity in the integration of (1.1) and, hence, the branch selection rule. For $J(r_0) < 1/4$, near the critical point we obtain

$$X_\pm = |\xi|^{1/2(1\pm\nu_0)}\exp\left[-\frac{i\pi}{2}(1\pm\nu_0)S(\xi)\right]\Phi_\pm,$$

$$S(\xi) = \begin{cases} 0 & \text{when } \xi > 0, \\ \pm 1 & \text{when } \xi < 0,\ U_0' \gtrless 0, \end{cases}$$

$$U_0 = V/r + kW/m \text{ when } r = r_0.$$

We distinguish a class of flows in which there are no SNM at internal points of the segments $[R_1R_2]$. We consider the Reynolds stresses $\tau = -\langle uv\rangle$, where $\langle uv\rangle$ is the average over z and φ. Since u and v have the form of normal waves, then $2\tau r = \mathrm{Re}(qv^*)\exp(2t\,\mathrm{Im}\,\omega)$. Whence, using (1.1), we can obtain

$$\frac{d}{dr}(\tau r^2) = \frac{\mathrm{Im}\,\omega}{2}\left\{-\left(\frac{A}{|\sigma|^2} + \frac{2G\,\mathrm{Re}\,\sigma}{|\sigma|^4}\right)|X|^2 + \frac{d}{dr}\left[\left(\frac{k^2}{m^2}\Omega r - \frac{k}{m}W'\right)\left|\frac{X}{\sigma}\right|^2\right]\right\}\exp(2t\,\mathrm{Im}\,\omega).$$

Thus, when Im $\omega \to 0$, τr^2 = const, except for possible discontinuities when $\sigma = 0$. Using the representation for X in the vicinity of the special point r_0 we find that when $J(r_0) < 1/4$

$$2\tau r^2 = \nu_0 m\,\mathrm{Im}\,\{AB^*\exp[-i\pi(1+\nu_0)S(\xi)]\}. \tag{2.2}$$

This expression describes the discontinuity of τ at the special point. In the considered case of flow between cylinders with a monotonic function U = V/r + kW/m we have $\tau = 0$ when $r = R_1, R_2$, and in the interval (R_1R_2) there cannot be more than one special point. Then there will be no discontinuity of τ at the special point. Whence it follows, in view of (2.2), that when $J_0 < 1/4$ either $X = X_+$, or $X = X_-$. When $J_0 \geqslant 1/4$, we can show, as in [5, 9], that there are no solutions of the SNM type. We write the conditions for absence of SNM for J > 0. They will be similar to those in [9]. Hence, we write only one of them and obtain the rest by analogy. It is necessary that throughout the flow region the inequalities $G > 0$, $G' \leqslant 0$, $U' > 0$, $U'' \geqslant 0$, $(\rho\Omega + k\rho rW'/m)' \geqslant 0$ are satisfied. We note the difference from [9]. For flows with round streamlines the inequality J > 0 is satisfied simultaneously at all k and m. Here there are always k/m for which the absence of SNM cannot be guaranteed.

The presented results in the case of $m^2 > 1$ can be directly extended to flows of the tornado type. Thus, the circle theorem is generalized by virtue of the fact that when $r \to 0$ the equality $q = O(r^2)$ is satisfied. For the problem with initial data, except $m^2 > 1$, we require that $v_0(0) = 0$.

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